

# On Combined Standard-Nonstandard or Hybrid $(q, h)$ -Deformations

B.L. Aneva<sup>a,1</sup>, D. Arnaudon<sup>b,2</sup>, A. Chakrabarti<sup>c,3</sup>,  
V.K. Dobrev<sup>d,a,4</sup> and S.G. Mihov<sup>a,5</sup>

<sup>a</sup> *Institute of Nuclear Research and Nuclear Energy  
Bulgarian Academy of Sciences  
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria* <sup>6</sup>

<sup>b</sup> *Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH  
CNRS, UMR 5108, associée à l'Université de Savoie  
LAPP, BP 110, F-74941 Annecy-le-Vieux Cedex, France*

<sup>c</sup> *Centre de Physique Théorique, CNRS UMR 7644  
Ecole Polytechnique, 91128 Palaiseau Cedex, France.*

<sup>d</sup> *School of Computing and Mathematics  
University of Northumbria  
Ellison Place, Newcastle upon Tyne, NE1 8ST, UK*

## Abstract

Combined  $(q, h)$ -deformations proposed by Kupershmidt and Ballesteros–Herranz–Parashar are studied. In each case a transformation is shown to lead to an equivalent, standard  $q$ -deformation. We briefly indicate that appropriate singular limits of the same type of transformations can however lead from standard biparametric  $(p, q)$ -deformations to non-hybrid but biparametric non-standard  $(g, h)$  ones. Finally a case of hybrid  $(q, h)$ -deformation is recalled, related to the superalgebra  $GL(1|1)$ .

INRNE-TH-00-02

LAPTH-800/00

S 080.0600

UNN-SCM-M-00-04

math.QA/0006206

June 2000

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<sup>1</sup>blan@inrne.bas.bg

<sup>2</sup>Daniel.Arnaudon@lapp.in2p3.fr

<sup>3</sup>chakra@cpht.polytechnique.fr

<sup>4</sup>vladimir.dobrev@unn.ac.uk, dobrev@inrne.bas.bg

<sup>5</sup>smikhov@inrne.bas.bg

<sup>6</sup>permanent address for V.K.D.

# 1 Introduction

Not only the standard  $q$ -deformation but also the nonstandard (Jordanian)  $h$ -deformation of  $GL(2)$  can be considered to be well-known. In each of these domains biparametric generalizations,  $(p, q)$  and  $(g, h)$  respectively, have been studied by number of authors. A large number of previous sources are cited in [1] and [2]. The dual quantum algebras of  $GL_{pq}$  and  $GL_{gh}$  were found in [3] and [4], respectively. Here we are concerned with certain proposals for combining these two distinct types into  $(q, h)$ -deformations. They will often be denoted as hybrid ones. In particular, we analyze the results of Kupershmidt [1] and of Ballesteros–Herranz–Parashar [2]. In each case, we show that a well-defined transformation eliminates  $h$  leaving a standard  $q$ -deformation. This transformation is not a arbitrary twist, but a straightforward similarity relation performed by a tensor square of an operator. This will be demonstrated explicitly in Sect. 2 and 3.

In Sect. 2 we start, in fact, with the 3-parameter  $(q, h, h')$  deformation of [1]. Already at this level we are able to construct a similarity transformation reducing the formalism to a 1-parameter deformation. The surviving single parameter  $q'$  is expressed explicitly as a function of  $(q, h, h')$ . For the case particularly advocated in [1], namely  $h' = 0$ , one has simply  $q' = q$ .

In Sect. 3 we start by transforming the  $R_{q,h}$  matrix of [2] to  $R_q$ . Then we show how their relevant results can be much better understood in the context of an explicitly presented, “coalgebra conserving” map. This illuminates several aspects and goes beyond the case of  $4 \times 4$  matrices.

In Sect. 4 we add some comments on maps and singular limits of transformations. Their nontrivial consequences [5, 6, 7, 8] are indicated by citing appropriate references. The passage from a standard biparametric  $(p, q)$ -deformation to a nonstandard  $(g, h)$  one is presented in this context.

In Sect. 5 we arrive (at last) to a hybrid  $(q, h)$  deformation where  $h$  cannot be transformed away. This turns out to be a hybrid deformation of the superalgebra  $GL(1|1)$ , already studied in [9]. It is located as the case  $R_{H1.2}$  in the classification of  $4 \times 4$   $R$ -matrices in [10], which we briefly recall.

Finally we would like to come back to Sections 2 and 3. Instead of briefly stating the equivalence  $(q, h) \longrightarrow (q)$ , we have chosen to present our elementary analysis explicitly and in some detail. We consider this worthwhile for dissipating some confusions. Several authors have presented attractive looking hybrid deformations without noticing disguised equivalences. We ourselves devoted time and effort to their study before reducing them to usual deformations. We hope that our analysis will create a more acute awareness of traps in this domain.

## 2 Kupershmidt's $(q, h, h')$ and $(q, h)$ deformations

We start by noting that the group relations given by the set of equations (5) of [1] can be written as

$$\begin{aligned}
 ca &= ac, & bd &= db \\
 cb &= qbc - hac - h'db \\
 ad &= da + (q-1)bc - hac - h'db \\
 qba &= ab + ha^2 + h'b^2 - h(da - bc) \\
 cd &= qdc - hc^2 - h'd^2 + h'(da - bc).
 \end{aligned} \tag{2.1}$$

From the second and the third equations of (2.1) one obtains

$$ad - qbc + hac + h'bd = da - bc = ad - cb. \tag{2.2}$$

Substituting from (2.2) the l.h.s. for  $(da - bc)$  in the fourth and fifth equations of (2.1) one gets back exactly (5) of [1]. Compared to his original version ours has the following advantages

- Adopting the ordering

$$d > a > b > c \tag{2.3}$$

all the terms in increasing order  $(ca, bd, \dots)$  are on the l.h.s. of (2.1), whereas the square terms and terms in decreasing order are on the r.h.s. This solves the ordering problem encountered in [1] when  $h'$  was taken different from 0.

- The roles of the parameters  $h, h'$  are now more simple and symmetrical. The terms bilinear in them (like  $hh'bd$  and  $h'^2bd$ ) do not appear in (2.1). The corresponding complementary (upper and lower triangular) linear contributions of  $h$  and  $h'$  in the  $R$ -matrix to be presented below correspond directly to this feature (the possibility of linearizing the contributions in (2.1)).
- The simpler form of (2.1) facilitates the construction of the  $R$ -matrix form of the RTT relations. This, in turn, facilitates the construction of the explicit similarity transformation leading to a 1-parametric equivalent deformation

$$(q, h, h') \longleftrightarrow q' \tag{2.4}$$

where  $q'$  is a specific function of  $(q, h, h')$  to be presented below. We could have derived the same final results using the more complicated version of (2.1) in [1]. But (2.1) is preferable.

## Solution of the RTT constraints

Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad T_1 = T \otimes 1, \quad T_2 = 1 \otimes T, \quad (2.5)$$

where  $a, b, c, d$  satisfy (2.1) and let  $R$  satisfy

$$RT_1T_2 = T_2T_1R. \quad (2.6)$$

Using (2.1) systematically, one obtains a solution involving an arbitrary parameter  $\kappa$ . It is

$$R = \begin{pmatrix} 1 & -h\kappa & h\kappa & 0 \\ 0 & q\kappa & (1 - q\kappa) & 0 \\ 0 & (1 - \kappa) & \kappa & 0 \\ 0 & -h'\kappa & h'\kappa & 1 \end{pmatrix}. \quad (2.7)$$

This does *not* satisfy the Yang–Baxter (YB) relations for all values of  $\kappa$ . In fact, in order that  $R$  satisfies YB the parameter  $\kappa$  must satisfy the following quadratic equation:

$$\kappa^2(q + hh') - \kappa(q + 1) + 1 = 0 \quad (2.8)$$

(Thus, for example, the particularly simple form for  $\kappa = 1$  does not satisfy YB unless  $hh' = 0$ .) The presence of  $\kappa$  at this stage permits the existence of two solutions satisfying YB constraints and related through  $R \longleftrightarrow R_{21}^{-1}$ .

We find it convenient for the similarity transformation to be introduced below to write down the two solutions of (2.8) in the following manner:

$$\kappa = \kappa_1 = (1 + \eta^{-1}h)^{-1}, \quad \kappa = \kappa_2 = (1 + \eta h')^{-1} \quad (2.9)$$

where the parameter  $\eta$  satisfies the quadratic

$$\eta^{-1}h + \eta h' = q - 1 \quad (2.10)$$

Our equation (2.10) is the same as the one used in [1] eq. (15) to eliminate  $h'$  at the level of the vector basis of the Poisson bracket algebra. But the rôle of our  $\eta$  (corresponding to  $t$  in [1]) is different. We continue to allow  $h'$  to be arbitrary and finally use a similarity transformation to arrive at an equivalent 1-parameter deformation.

## Similarity transformation to a 1-parameter deformation

Define

$$G = \begin{pmatrix} 1 & \eta \\ \zeta & (1 + \eta\zeta) \end{pmatrix} \quad G^{-1} = \begin{pmatrix} (1 + \eta\zeta) & -\eta \\ -\zeta & 1 \end{pmatrix} \quad (2.11)$$

where the parameters  $\eta, \zeta$  are given by

$$h = (q - 1) \frac{\eta(1 + \eta\zeta)}{1 + 2\eta\zeta}, \quad h' = (q - 1) \frac{\eta}{1 + 2\eta\zeta}, \quad (2.12)$$

or

$$\begin{aligned} \eta &= (2h')^{-1}((q - 1) \pm \lambda), & \eta^{-1} &= (2h)^{-1}((q - 1) \mp \lambda) \\ \zeta &= \mp h' \lambda^{-1}, & \lambda &= \sqrt{(q - 1)^2 - 4hh'} \end{aligned} \quad (2.13)$$

(For  $\lambda = 0$ ,  $\zeta$  diverges. This point should be approached as a limit after transforming.)

One obtains, after subtle simplifications

$$R' = (G \otimes G)R(G^{-1} \otimes G^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa/\kappa_1 & (1 - \kappa/\kappa_1) & 0 \\ 0 & (1 - \kappa/\kappa_2) & \kappa/\kappa_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.14)$$

where  $\kappa_1, \kappa_2$  are given by (2.9). (In the light of the remarks following (2.13), note that  $\zeta$  does not appear in  $R'$ , only  $\eta$  through  $\kappa_1$  and  $\kappa_2$ . Reality restrictions are discussed at the end of this section.)

Now the statement leading to (2.9) is evident: for  $\kappa = \kappa_1$  one gets

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (1 - \kappa_1/\kappa_2) & \kappa_1/\kappa_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.15)$$

the standard lower triangular form of the YB solution for the single parameter  $\kappa_1/\kappa_2$ . Similarly, for  $\kappa = \kappa_2$ , one gets

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa_2/\kappa_1 & (1 - \kappa_2/\kappa_1) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.16)$$

the standard upper-triangular YB solution for the single parameter  $\kappa_2/\kappa_1$ . In fact (2.15) and (2.16) are related as the pair  $R', (R'_{21})^{-1}$ .

Let us now define

$$T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \quad (2.17)$$

such that it satisfies

$$R'T'_1T'_2 = T'_2T'_1R'. \quad (2.18)$$

The group relations turn out to be independent of the parameter  $\kappa$  of (2.14) (just as (2.1) are independent of  $\kappa$  in (2.7)). One obtains

$$\begin{aligned} c'a' &= a'c', & b'd' &= d'b', & c'b' &= q'b'c', \\ a'd' &= d'a' + (q' - 1)b'c', & q'b'a' &= a'b', & c'd' &= q'd'c', \end{aligned} \quad (2.19)$$

where we have set

$$q' = \frac{\kappa_2}{\kappa_1} = \frac{1 + \eta^{-1}h}{1 + \eta h'} \quad (2.20)$$

with  $\eta^{-1}h + \eta h' = q - 1$ . Note that

$$\begin{aligned} \text{for } h' = 0, & \quad q' = q \\ \text{for } h = 0, & \quad q' = q^{-1} \end{aligned} \quad (2.21)$$

A more complete discussion of the domains of  $q'$  follows below.

Setting  $h = h' = 0$  in (2.1) and adding primes one obtains (2.19).

The relations (2.19) can also be obtained from (2.1) by transforming with  $G$ . We have preferred to construct the corresponding  $R$ -matrices first from their intrinsic interest and also for elucidating the significance of the free parameter  $\kappa$  arising at the RTT level before imposing the YB constraints.

In [1] after setting  $h' = 0$  the  $(q, h)$  deformed system is reformulated using a certain ordering. Our preceding study contains the  $R$ -matrix for this case as the particular one obtained by setting in (2.7)

$$\begin{aligned} \kappa &= \kappa_2 = (1 + \eta h')^{-1} = 1 & (\text{for } h' = 0) \\ \kappa_1 &= (1 + \eta^{-1}h)^{-1} = q^{-1} \end{aligned} \quad (2.22)$$

giving

$$R' = \begin{pmatrix} 1 & -h & h & 0 \\ 0 & q & (1 - q) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.23)$$

The transformation to 1-parameter form is now by

$$G = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \quad \text{with } \zeta = 0, \eta = \frac{h}{q-1} \quad (2.24)$$

in (2.11). The upper triangular form (2.16) now becomes

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & (1-q) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.25)$$

The transformed  $(a, b, c, d)$ , namely  $(a', b', c', d')$  now satisfies (2.19) with  $q' = q$ .

The results for  $h' = 0$  can be obtained directly by starting with a  $(q, h)$  system given by (16) of [1] with one necessary correction. The group relations should be written as

$$\begin{aligned} ba &= q^{-1}a(b+ha) - q^{-1}hda + q^{-2}hc(b+ha), \\ bd &= db, \\ bc &= q^{-1}c(b+ha), \\ ad &= da + (1-q^{-1})cb - q^{-1}hca, \\ ac &= ca, \\ dc &= q^{-1}c(d+hc). \end{aligned} \quad (2.26)$$

(The last two terms of the first equation has each an extra factor  $q^{-1}$  as compared to (16) of [1]. Setting  $h' = 0$  in (5) of [1] and reordering one indeed gets our version. Starting directly with (2.26), the RTT relations and the YB constraints can indeed be shown to lead to (2.23). Then (2.24) eliminates  $h$  leading to (2.25) and to (2.19) with  $q' = q$ . The case  $h = 0$  can be treated quite analogously setting (taking the lower sign in (2.13) for  $\eta, \zeta$ )

$$\eta = 0, \quad G = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}, \quad \zeta = \frac{h'}{q-1}. \quad (2.27)$$

In [1]  $h'$  was eliminated at the stage of Poisson brackets and  $h$  was retained. It was assumed that one thus obtains an authentic 2-parameter  $(q, h)$ -deformation. We have shown that this is not the case. Our transformation for the original  $(q, h, h')$  case shows that, from the start, one has always been dealing with a heavily disguised  $q$ -deformation. This statement should however be qualified by taking a closer look at different domains of the parameter space of  $(q, h, h')$ . We consider below real values of  $(q, h, h')$ .

For both  $(h, h') \neq 0$  (the cases  $h' = 0$  and  $h = 0$ , with  $q' = q$  and  $q' = q^{-1}$  respectively, can be considered simply and analogously), from (2.20),

$$q' = \frac{q+1 \pm \sqrt{(q-1)^2 - 4hh'}}{q+1 \mp \sqrt{(q-1)^2 - 4hh'}} \quad (2.28)$$

Apart from the very special case

$$q = -1 \quad q' = -1 \quad (2.29)$$

we note that more generally *i*) for any  $q$  and  $hh' < 0$  and *ii*) for  $(q-1)^2 > 4hh' > 0$ ,  $q'$  is always real. We consider this as the generic case.

Another very special case is (for  $hh' > 0$ )  $|q-1| = 2\sqrt{hh'}$  when  $q' = 1$ . See the remark following (2.13) concerning this singular point. Here one has a classical solution with commuting  $a', b', c', d'$  !

For  $hh' > 0$  and  $|q-1| < 2\sqrt{hh'}$ ,  $q' = e^{\pm i\delta}$ , a complex phase. Here  $q'$  can even be a root of unity. Thus starting from a complex deformation, one can obtain by the transformation with  $G$  an equivalent deformation with 3 real parameters related through (2.28).

### 3 The Ballesteros–Herranz–Parashar (BHP) case

#### 3.1 Transformation to $R_q$

The BHP two-parametric deformation (Sect. 4 of [2]) leads to the  $R$ -matrix

$$R_{q,h} = \begin{pmatrix} 1 & h & -qh & h^2 \\ 0 & q & 1-q^2 & qh \\ 0 & 0 & q & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.1)$$

The authors present it as a superposition of standard ( $q$ ) and nonstandard ( $h$ ) deformations. It has indeed the attractive property that for  $h = 0$  and  $q = 1$  respectively one obtains the standard  $R_q$  and the nonstandard  $R_h$  matrices.

Now consider a similarity transformation of  $R_{q,h}$  by  $M \otimes M$  where

$$M = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \quad M^{-1} = x^{-1} \begin{pmatrix} 1 & -y \\ 0 & x \end{pmatrix} \quad (3.2)$$

with  $y = \frac{h}{q-1}$  ( $h \neq 0$ ,  $q \neq 1$ ) and  $x$  is an arbitrary nonzero parameter. In the notation of [2]

$$y = \frac{a_+}{2a} \quad (q = e^a) \quad (3.3)$$

One obtains, independently of the choice of  $x$ ,

$$(M^{-1} \otimes M^{-1})R_{q,h}(M \otimes M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_q. \quad (3.4)$$

Thus it is seen that  $h$  can be transformed away.



### 3.2 A coalgebra preserving map

The significance of this equivalence (and that of the related Hopf algebraic results of [2]) are better understood in the context of a simple class of coalgebra preserving maps, presented below. (The content of the mapping can be considered, in a certain sense, to be trivial. Elucidating this aspect is precisely our purpose.) These maps can be generalized to higher dimensional algebras. But we here consider only  $\mathcal{U}_q(gl(2))$ .

One starts with the standard  $\mathcal{U}_q(sl(2))$  algebra

$$\begin{aligned} [J_0, J_{\pm}] &= \pm 2J_{\pm} & (q^{J_0} J_{\pm} &= q^{\pm 2} J_{\pm} q^{J_0}) \\ [J_+, J_-] &= \frac{q^{J_0} - q^{-J_0}}{q - q^{-1}} \equiv [J_0] \end{aligned} \quad (3.5)$$

with the coalgebra structure

$$\begin{aligned} \Delta(J_0) &= J_0 \otimes 1 + 1 \otimes J_0 & (\Delta(q^{J_0}) &= q^{J_0} \otimes q^{J_0}) \\ \Delta(J_{\pm}) &= q^{J_0/2} \otimes J_{\pm} + J_{\pm} \otimes q^{-J_0/2} \\ S(q^{J_0/2}) &= q^{-J_0/2} \\ S(J_{\pm}) &= -q^{-J_0/2} J_{\pm} q^{J_0/2} \\ \epsilon(q^{J_0/2}) &= 1, & \epsilon(J_{\pm}) &= 0 \end{aligned} \quad (3.6)$$

(Note that for the antipode  $S$ , we have not replaced  $q^{-J_0/2} J_{\pm} q^{J_0/2}$  by  $q^{\mp 1} J_{\pm}$ . This last form uses (3.5) which will be modified by the map, keeping the structure (3.6) intact in terms of the new generators.)

Next one sets

$$\begin{aligned} J'_0 &= J_0 \\ J'_+ &= J_+ \\ J'_- &= b_1 J_- + b_2 (q^{J_0/2} - q^{-J_0/2}) + b_3 J_+ . \end{aligned} \quad (3.7)$$

Choosing (with  $q = e^a$ )

$$b_1 = \frac{\sinh(a)}{a}, \quad b_2 = -\frac{a_+}{2a^2} \quad b_3 = -\frac{a_+^2}{4a^2} \quad (3.8)$$

one obtains the case (4.6) of [2] (our  $(J'_0, J'_+, J'_-)$  corresponding to their  $(J'_3, J_+, J_-)$ ). Indeed

$$\begin{aligned} [J'_0, J'_+] &= 2J'_+ \\ [J'_0, J'_-] &= -2J'_- - \frac{a_+}{a} \frac{\sinh(aJ'_0/2)}{a/2} - \frac{a_+^2}{a^2} J'_+ \\ [J'_+, J'_-] &= \frac{\sinh(aJ'_0)}{a} + \frac{a_+}{a} \frac{e^a - 1}{2a} (e^{-aJ'_0/2} J'_+ + J'_+ e^{aJ'_0/2}) \end{aligned} \quad (3.9)$$

Moreover, the coalgebra structure induced by this map has the same expression in terms of the new generators, i.e.

$$\begin{aligned}\Delta(J'_0) &= J'_0 \otimes 1 + 1 \otimes J'_0 \\ \Delta(J'_\pm) &= q^{J'_0/2} \otimes J'_\pm + J'_\pm \otimes q^{-J'_0/2}\end{aligned}\tag{3.10}$$

and so on. This is achieved under the single condition that the coefficients of  $q^{\pm J_0}$  in (3.7) are opposite. (Note that the latter statement would be true also if, in addition, we write  $J'_+ = a_1 J_+ + a_2(q^{J_0/2} - q^{-J_0/2}) + a_3 J_-$ .) The whole Hopf algebra described by  $(J'_0, J'_+, J'_-)$  then reproduces that of BHP. This means that the BHP Hopf algebra is equivalent to  $\mathcal{U}_q(gl(2))$ .

## 4 Comments on singular limits of transformations

Up to now we have been considering regular, invertible transformations making evident the trivial nature of the passage

$$R_q \longleftrightarrow R_{q,h} .\tag{4.1}$$

The map (3.7) is consistent with this due to the conservation of the structure of the coalgebra. When a map has to be followed by a twist [11, 12] to arrive at a sought for coalgebra, the situation can be of interest. It has been shown elsewhere [5] how the universal  $\mathcal{R}_h$  matrix (introduced first in [13]) can be obtained, through a twist, starting from the trivial classical one ( $\mathcal{U}_h(sl(2))$  is a triangular Hopf algebra). Another interesting possibility is the use of a transformation singular in the limit  $q \rightarrow 1$  but in such a specific fashion that ( $G(q, h)$  being singular at  $q = 1$ ),

$$(G(q, h)^{-1} \otimes G(q, h)^{-1} R_q G(q, h) \otimes G(q, h)) \Big|_{q=1} = R_h .\tag{4.2}$$

In contrast to (4.1) this passage is non-invertible and the end-product is not a hybrid  $R_{q,h}$  but a nonstandard  $R_h$ . This can be considered as an operator equation between universal R-matrices and  $G$  given by (as shown in [6]):

$$G(q, h) = E_q(\eta J_+) \quad \text{with} \quad \eta = \frac{h}{q-1}\tag{4.3}$$

$$\text{and} \quad E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}, \quad [n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}\tag{4.4}$$

Alternatively (4.2) can be regarded as a matrix equation implementing  $j_1 \otimes j_2$  representations. This technique can be generalized to  $GL(N)_q$  [6] and also to obtain nonstandard quasi-Hopf algebras [7].

Note that in (4.2)  $\eta$  has the same form as in (2.11) or (3.2). But the crucial difference is that one takes the limit  $q \rightarrow 1$ .

In the above-mentioned references universal  $\mathcal{R}$  matrices have been studied but only for  $R_q$  and  $R_h$ . Here, in conclusion, we indicate how one can treat the biparametric case involving  $R_{p,q}$  and  $R_{g,h}$ . (Note that no hybrid deformation is involved here.) We restrict the study to the fundamental case of  $4 \times 4$  matrices.

We start with

$$R_{pq} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & pq & p-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad (4.5)$$

and define (with  $j_1 \otimes j_2 = \frac{1}{2} \otimes \frac{1}{2}$  in (4.3))

$$G = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \quad (4.6)$$

but now  $\eta$  will be chosen differently. One obtains

$$(G^{-1} \otimes G^{-1})R_{p,q}(G \otimes G) = \begin{pmatrix} p & p(1-q)\eta & (q-1)\eta & (1-p)(q-1)\eta^2 \\ 0 & pq & p-q & q(p-1)\eta \\ 0 & 0 & 1 & (1-p)\eta \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (4.7)$$

Now let  $q \rightarrow 1$  and  $p \rightarrow 1$  in such a fashion that

$$\left(\frac{q-1}{p-1}\right)^{1/2} = \lambda = \text{constant}. \quad (4.8)$$

Set

$$\eta = \frac{\eta_0}{((p-1)(q-1))^{1/2}}, \quad \lambda\eta_0 = h, \quad \lambda^{-1}\eta_0 = -g. \quad (4.9)$$

Then

$$((G^{-1} \otimes G^{-1})R_{p,q}(G \otimes G))|_{(p \rightarrow 1, q \rightarrow 1)} = \begin{pmatrix} 1 & -h & h & gh \\ 0 & 1 & 0 & -g \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.10)$$

Thus we obtained the 2-parametric nonstandard  $R$ -matrix. For  $g = h$  and  $g = 0$ , we obtain the two known forms of  $R_h$ . The 2-parametric universal coloured, nonstandard  $R$ -matrix for deformed  $gl(2)$  is obtained in Sect. 3 of [8] implementing a twist. Here we presented the  $4 \times 4$  case to show how  $\eta$  is quite simply modified from (4.3) to (4.9) as one passes from the 1- to the 2-parametric case.

## 5 An authentic hybrid $(q, h)$ deformation: $GL_{q,h}(1|1)$

In Sect. 2 and Sect. 3 we have shown that the hybrid  $(q, h)$  deformations in [1] and [2] are in fact disguised  $q$ -ones. In the search of hybrid deformation we also check with the classification of  $4 \times 4$   $R$ -matrices in [10]. There we find seven triangular cases:

$$R_{S2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & p & 1-pq & 0 \\ & & q & 0 \\ & & & 1 \end{pmatrix} \quad (5.1)$$

$$R_{S2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & p & 1-pq & 0 \\ & & q & 0 \\ & & & -pq \end{pmatrix} \quad (5.2)$$

$$(R_{H1,3})_{|k=1, p=-h, q=-g} = \begin{pmatrix} 1 & -h & h & gh \\ & 1 & 0 & -g \\ & & 1 & g \\ & & & 1 \end{pmatrix} \quad (5.3)$$

$$(R_{H2,3})_{|k=1, p=x_1, q=x_2, s=x_3} = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & 0 & x_2 \\ & & 1 & x_1 \\ & & & 1 \end{pmatrix} \quad (5.4)$$

$$R_{S0,1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad (5.5)$$

$$R_{S0,2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & -1 & 0 & 0 \\ & & -1 & 0 \\ & & & 1 \end{pmatrix} \quad (5.6)$$

$$R_{q,h} = (R_{H1,2})|_{p=1,k=h} = \begin{pmatrix} 1 & 0 & 0 & h \\ & 1 & 1-q & 0 \\ & & q & 0 \\ & & & -q \end{pmatrix} \quad (5.7)$$

Note that in [10] the  $R$ -matrices are given in two versions: homogeneous  $R_{H\dots}$  and scaled  $R_{S\dots}$ . The scaled versions are simpler but in some cases, in order not to lose some symmetry among the parameters we use the homogeneous versions with only an overall rescaling.

The case  $R_{S2,1}$  is the 2-parameter  $p, q$  deformation,  $GL_{pq}(2)$  the dual of which is given in [3]. The case  $R_{S2,2}$  is a superalgebra - the known  $p, q$  deformation of  $GL_{pq}(1|1)$ , the dual of which is given in [14, 15, 16].

*Remark:* Note that here (and below for  $R_{q,h}$ ) we consider ordinary, not graded,  $R$ -matrices. The results can be translated to the graded formalism. There is a one-to-one correspondence between the results obtained through the two approaches (which was noticed first for solutions of YBE and graded YBE in [17]). This correspondence may be given also through “transmutation” in the sense of [18]. This aspect is considered in the context of  $sl(1|2)$  in [19]. In our paper the superalgebraic aspect becomes evident after implementation of the RTT formalism.

In the third case we have written the homogeneous version  $R_{H1,3}$  of [10] with  $k = 1$  and renamed parameters. This seems the natural scaling (and not the  $R_{S1,3}$ ). The result is indeed with 2 parameters, i.e., this is the 2-parameter Jordanian  $GL_{gh}(2)$ . The dual was found in [4].

In the fourth case we have written the homogeneous version  $R_{H2,3}$  of [10] with  $k = 1$  and renamed parameters. From the RTT relations we obtain the following:

$$\begin{aligned} acx_1 + cax_2 + c^2x_3 &= 0 \\ ac - ca + c^2x_2 &= 0 \\ -ac + ca + c^2x_1 &= 0 \\ -c^2x_1 + cd - dc &= 0 \\ -c^2x_2 - cd + dc &= 0 \\ c^2x_3 + cdx_1 + dcx_2 &= 0 \\ ad - cax_1 + cdx_2 - da &= 0 \\ -acx_2 - ad + da + dcx_1 &= 0 \\ bc - cax_2 - cb + dcx_2 &= 0 \\ -acx_1 - bc + cb + cdx_1 &= 0 \end{aligned}$$

$$\begin{aligned}
& -a^2x_1 + ab + adx_1 - ba + cbx_2 + cdx_3 = 0 \\
& -a^2x_2 - ab + ba + bcx_1 + dax_2 + dcx_3 = 0 \\
& bd - cax_3 - cbx_1 - dax_2 - db + d^2x_2 = 0 \\
& -acx_3 - adx_1 - bcx_2 - bd + db + d^2x_1 = 0 \\
& -a^2x_3 - abx_1 - bax_2 + bdx_1 + dbx_2 + d^2x_3 = 0
\end{aligned} \tag{5.8}$$

It is necessary to consider several cases.

1. In the case  $x_1 + x_2 \neq 0$  (and arbitrary  $x_3$ ) from the above follow:

$$\begin{aligned}
c^2 &= 0, & ca &= ac = 0, & dc &= cd = 0, \\
da &= ad, & cb &= bc, \\
a^2 &= d^2 = ad + bc \\
ab &= bd = ba + (x_1 - x_2)bc, & db &= bd + (x_2 - x_1)bc
\end{aligned} \tag{5.9}$$

These relations make the resulting algebra rather degenerate. Moreover, in order to build a PBW basis we have to look also for higher order relations. For instance, using these relations we obtain:

$$a^3 = a^2d + abc = a^2d$$

Furthermore one can eliminate  $bc = a^2 - ad$ , and  $bd = ba + (x_1 - x_2)(a^2 - ad)$ . From all these follows that the PBW basis may have only the following monomials:

$$b^n a^k, \quad a^\ell d, \quad c, \quad n, k \in \mathbb{Z}_+, \quad \ell = 0, 1. \tag{5.10}$$

2. Next we consider the case  $x_1 = -x_2 = h$ ,  $x_3 \neq -h^2$  then from (5.8) follow:

$$\begin{aligned}
c^2 &= 0, & ca &= ac = 0, & dc &= cd = 0, \\
da &= ad, & cb &= bc, & a^2 &= d^2 \\
ab &= ba + h(a^2 + bc - ad), & db &= bd - h(a^2 + bc - ad).
\end{aligned} \tag{5.11}$$

The resulting algebra is also degenerate, though the above relations are less restrictive than the previous case and the possible PBW basis is richer:

$$b^n a^k d^\ell, \quad b^n c^\ell, \quad n, k \in \mathbb{Z}_+, \quad \ell = 0, 1. \tag{5.12}$$

3. Finally in the case  $x_1 = -x_2 = h$ ,  $x_3 = -h^2$  this coincides with the case  $R_{H1,3}$  when  $g = -h$ .

The fifth case  $R_{S0,1}$  is a special case of  $R_{H2,3}$  when  $x_1 = x_2 = 0$ ,  $x_3 = 1$ . Thus, the resulting algebra relations are obtained from (5.11) setting  $h = 0$ .

In the sixth case  $R_{S0,2}$  the RTT relations give:

$$\begin{aligned}
c^2 &= 0, & ca &= ac = 0, & dc &= cd = 0, \\
da &= ad, & cb &= bc, & a^2 &= d^2 \\
ab + ba &= 0, & db + bd &= 0
\end{aligned} \tag{5.13}$$

This is a superalgebra, also degenerate like the previous two cases. The PBW basis would be as in (5.12).

Finally, we are left with the seventh case, which we have anticipated to be a hybrid one by the notation  $R_{q,h}$ . Note that setting  $q = 1$ ,

$$R_h = R_{1,h} = \begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (5.14)$$

still depends on  $h$  and is triangular (in the sense  $R_{21} = R^{-1}$ ).

Now we obtain the RTT relations by implementing (5.7) in (2.6) with (2.5):

$$\begin{aligned} ba &= ab + hcd, & ca &= q^{-1}ac, \\ da &= ad - (1 - q^{-1})bc, & cb &= q^{-1}bc, \\ db &= -bd + hq^{-1}ac, & dc &= -qcd, \\ c^2 &= 0, & ha^2 &= (q + 1)b^2 + hd^2. \end{aligned} \quad (5.15)$$

This is a superalgebra which we shall denote by  $GL_{q,h}(1|1)$ . It was first written in [9], where also the dual quantum algebra was given. This is indeed a hybrid deformation of  $GL(1|1)$  since it is known from [10] that no transformation of the form

$$R_{qh} \longrightarrow (M \otimes M)R_{qh}(M^{-1} \otimes M^{-1}) \quad (5.16)$$

can lead to  $R_q$ .

Using the transformation (5.16) with

$$M = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \quad (5.17)$$

one gets

$$R_{q,h} = \begin{pmatrix} 1 & 0 & 0 & hx^4 - (1+q)(xy)^2 \\ 0 & 1 & 1-q & -2xy \\ 0 & 0 & q & -2qxy \\ 0 & 0 & 0 & -q \end{pmatrix}. \quad (5.18)$$

For no choice of  $x, y$  one obtains for the transformed  $R$  the form  $R_{q,h=0}$ , in contrast with the results of Sect. 2 and 3.

For  $1 + q \neq 0$  one may choose to eliminate the top right hand element by setting  $y = \pm x \left( \frac{h}{1+q} \right)^{1/2}$ . Denoting  $h' = -2xy$ ,

$$R_{q,h} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1-q & h' \\ 0 & 0 & q & qh' \\ 0 & 0 & 0 & -q \end{pmatrix}. \quad (5.19)$$

In conclusion we repeat that in order to display the sharp contrast between the mixed deformations of Sect. 2 and 3 and the present case we have restricted our considerations in both cases to similarity transformations, i.e. coboundary twists  $\mathcal{F} \equiv (g^{-1} \otimes g^{-1})\Delta(g)$ , which transform  $\mathcal{R}$  matrices as  $\mathcal{R}^{\mathcal{F}} \equiv \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1} = (g^{-1} \otimes g^{-1})\mathcal{R}(g \otimes g)$ . Other interesting aspects can be explored by implementing more general twists. We refer to the discussion in [20] concerning the three deformed versions of  $gl(1|1)$ .

**Acknowledgments:** One of us (A.C.) acknowledges with pleasure interesting discussions with Petr Kulish. V.K.D. would like to thank for hospitality CERN-TH, where part of the work was done. This work was supported in part by the CNRS-BAS France/Bulgaria agreement number 6608.

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